q-Analogs of some congruences involving Catalan numbers

Roberto Tauraso
Dipartimento di Matematica
Università di Roma "Tor Vergata", Italy
tauraso@mat.uniroma2.it
http://www.mat.uniroma2.it/∼tauraso

Abstract

We provide some variations on the Greene-Krammer's identity which involve q-Catalan numbers. Our method reveals a curious analogy between these new identities and some congruences modulo a prime.

1 Introduction

In [9], John Greene proved the following conjecture made by Daan Krammer

$$1+2\sum_{k=1}^{n-1}(-1)^k q^{-\binom{k}{2}}\begin{bmatrix}2k-1\\ k\end{bmatrix}_q=\begin{cases}\left(\dfrac{m}{5}\right)\sqrt{5} & \text{if } 5 \mid n\\[2ex] \left(\dfrac{n}{5}\right) & \text{otherwise}\end{cases}$$

where $q=e^{2\pi m i/n}$ with $\gcd(n,m)=1$ and $\left(\frac{n}{p}\right)$ is the standard Legendre symbol (see also [2] and [6]). On the other hand if we take $q=1$, and let n be a power of a prime p then the l.h.s satisfies the following congruence which appears in [13]

$$1+2\sum_{k=1}^{p^a-1}(-1)^k\binom{2k-1}{k}=\sum_{k=0}^{p^a-1}(-1)^k\binom{2k}{k}\equiv\left(\frac{p^a}{5}\right)\pmod{p}.$$

In this note we would like to present more examples of the same flavour involving the q-Catalan numbers.

2 Notations and properties of q-binomial coefficients

The *Gaussian q-binomial coefficient* $\begin{bmatrix}n\\ k\end{bmatrix}_q$ is defined as

$$\begin{bmatrix}n\\ k\end{bmatrix}_q=\begin{cases}(q;q)_n(q;q)_k^{-1}(q;q)_{n-k}^{-1} & \text{if } 0\le k\le n\\ 0 & \text{otherwise}\end{cases}$$

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where $(z;q)_n = \prod_{j=0}^{m-1}(1 - zq)$. It is a polynomial in q which satisfies the following relations for $0 \le k \le n$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \tag{2.1}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \tag{2.2}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{1/q}. \tag{2.3}$$

We define the q-Fibonacci polynomials ([3]) by the recursion

$$F_n^q(t) = F_{n-1}^q(t) + q^{n-2} t F_{n-2}^q(t)$$

with initial values $F_0^q(t) = 0$, $F_1^q(t) = 1$. The following identity yields an explicit formula

$$F_n^q(t) = \sum_{k \ge 0} q^{k^2} \begin{bmatrix} n-1-k \\ k \end{bmatrix}_q t^k. \tag{2.4}$$

There are various q-analogs of the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ (see for example [8]). We will consider the following definition

$$C_n^q = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q = \begin{bmatrix} 2n \\ n \end{bmatrix}_q - q \begin{bmatrix} 2n \\ n+1 \end{bmatrix}_q.$$

where $[n+1]_q = (1-q)/(1-q^{n+1})$. By [5], C_n^q is a polynomial with respect to q.

3 q-Binomial coefficient congruences

Let $\Phi_n(q)$ be the n-cyclotomic polynomial:

$$\Phi_n(q) = \prod_{\substack{0 \le m < n \\ \gcd(m,n)=1}} (q - e^{2\pi m i/n}).$$

We now deduce some properties that we will need later.

Lemma 3.1. *For $n > 1$*

$$\Phi_n(1) = \begin{cases} p & \text{if } n \text{ is a power of a prime } p \\ 1 & \text{otherwise} \end{cases}. \tag{3.1}$$

Proof. See for example [10] at page 160. □

Lemma 3.2. *For any positive integer a*

$$\begin{bmatrix} an \\ k \end{bmatrix}_q \equiv \begin{cases} \binom{a}{k/n} & \text{if } n|k \\ 0 & \text{otherwise} \end{cases} \pmod{\Phi_n(q)} \tag{3.2}$$

and

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \equiv \begin{cases} 1 & \text{if } k = 0, 1, n, n+1 \\ \\ 0 & \text{otherwise} \end{cases} \pmod{\Phi_n(q)}. \tag{3.3}$$

Proof. By [5], $\Phi_n(q)$ is a factor of $\begin{bmatrix} m \\ k \end{bmatrix}_q$ if and only if $\{k/n\} > \{m/n\}$ where $\{x\}$ denote the fractional part of x, namely $\{x\} = x - \lfloor x \rfloor$. Morever, by [4],

$$\begin{bmatrix} an \\ bn \end{bmatrix}_q \equiv \binom{a}{b} \pmod{\Phi_n(q)}.$$

□

Lemma 3.3. *The following congruences hold:*
for $k = 1, \ldots, n-1$

$$\begin{bmatrix} 2k-1 \\ k \end{bmatrix}_q \equiv (-1)^k q^{\frac{3k^2-k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \pmod{\Phi_n(q)}, \tag{3.4}$$

for $k = 0, \ldots, n-1$

$$\begin{bmatrix} 2k \\ k \end{bmatrix}_q \equiv (-1)^k q^{\frac{3k^2+k}{2}} \begin{bmatrix} n-1-k \\ k \end{bmatrix}_q \pmod{\Phi_n(q)}, \tag{3.5}$$

and

$$\begin{bmatrix} 2k \\ k+1 \end{bmatrix}_q \equiv \begin{cases} (-1)^{k+1} q^{\frac{3k^2+3k}{2}} \begin{bmatrix} n-k \\ k+1 \end{bmatrix}_q & \text{if } k = 0, 1, \ldots, n-2 \\ \\ 1 & \text{if } k = n-1 \end{cases} \pmod{\Phi_n(q)}. \tag{3.6}$$

Proof. Let $q = e^{2\pi m i/n}$ with $\gcd(n, m) = 1$. Since $q^k \neq 1$ for $k = 1, \ldots, n-1$ and $q^n = 1$, then

$$\begin{aligned} \begin{bmatrix} 2k-1 \\ k \end{bmatrix}_q &= \frac{(1-q^{2k-1}) \cdots (1-q^k)}{(1-q^k) \cdots (1-q)} \\ &= (-1)^k q^{\frac{3k^2-k}{2}} \frac{(1-q^{n-(2k-1)}) \cdots (1-q^{n-k})}{(1-q^k) \cdots (1-q)} \\ &= (-1)^k q^{\frac{3k^2-k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q . \end{aligned}$$

Hence

$$\begin{bmatrix} 2k-1 \\ k \end{bmatrix}_q - (-1)^k q^{\frac{3k^2-k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q$$

is a the polynomial in q which has at least the same roots of $\Phi_n(q)$ and the proof of (3.4) is complete. In a similar way we show the other two congruences (3.5) and (3.6). □

4 q-Identities

A fundamental result that we are going to use is the finite form of the Rogers-Ramanujan identities (see for example [1] p. 50): for $a \in \{0,1\}$

$$F^q_{n+1-a}(q^a) = \sum_{k\geq 0} q^{k^2+ak} \begin{bmatrix} n-a-k \\ k \end{bmatrix}_q = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j(5j+1-4a)}{2}} \begin{bmatrix} n \\ \lfloor \frac{n+2a-5j}{2} \rfloor \end{bmatrix}_q \tag{4.1}$$

The next q-identity has been proved in [7] with a computer proof. Here we show that the identity holds by using only some basic properties of q-binomial coefficients.

Theorem 4.1.

$$\sum_{k\geq 0} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q = (-1)^n \left(\frac{n+1}{3}\right) q^{\frac{1}{3}\binom{n}{2}} \tag{4.2}$$

Proof. Let the l.h.s. be $G(n)$ and let the r.h.s be $H(n)$. It's easy to verify that $G(n) = H(n)$ for $n = 0, 1, 2, 3$. Moreover for $n \geq 1$

$$H(n+3) = -(-1)^{n+1} \left(\frac{n+1}{3}\right) q^{\frac{1}{3}\binom{n}{2}+n+1} = -q^{n+1} H(n).$$

So it suffices to show that the same identity holds also for $G(n)$. By (2.1), we have that

$$\begin{aligned}
G(n+3) &= 1 - \sum_{k\geq 1} (-1)^{k-1} q^{\binom{k}{2}+n+3-2k} \begin{bmatrix} n+2-k \\ k-1 \end{bmatrix}_q + \sum_{k\geq 1} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n+2-k \\ k \end{bmatrix}_q \\
&= G(n+2) - q^{n+1} \sum_{k\geq 1} (-1)^{k-1} q^{\binom{k-1}{2}-(k-1)} \begin{bmatrix} n+1-(k-1) \\ k-1 \end{bmatrix}_q \\
&= G(n+2) - q^{n+1} \sum_{k\geq 0} (-1)^k q^{\binom{k}{2}-k} \begin{bmatrix} n+1-k \\ k \end{bmatrix}_q .
\end{aligned}$$

Moreover, by (2.2),

$$\begin{aligned}
\sum_{k\geq 0} (-1)^k q^{\binom{k}{2}-k} \begin{bmatrix} n+1-k \\ k \end{bmatrix}_q &= 1 - \sum_{k\geq 1} (-1)^{k-1} q^{\binom{k-1}{2}-1} \begin{bmatrix} n-k \\ k-1 \end{bmatrix}_q + \sum_{k\geq 1} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \\
&= -q^{-1} G(n-1) + G(n).
\end{aligned}$$

Thus, since by the induction hypothesis $G(n+2) = -q^n G(n-1)$ then

$$G(n+3) = G(n+2) - q^{n+1}(-q^{-1} G(n-1) + G(n)) = -q^{n+1} G(n).$$

$\square$

The last q-identity seems to be new. It is a q-analogue of a binomial identity contained in [13] and it has been conjectured for $d = 0$ by Z. W. Sun. Since the proof is rather technical we postpone it to the last section.

Theorem 4.2. *For $n \geq |d|$ then*

$$\sum_{k=0}^{n-1} q^k \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q = \sum_{k=0}^{n-|d|} q^{\frac{1}{3}\left(2(n-k)^2-(n-k)\left(\frac{n-|d|-k}{3}\right)-2d^2-1\right)} \left(\frac{n-d-k}{3}\right) \begin{bmatrix} 2n \\ k \end{bmatrix}_q \tag{4.3}$$

The p-congruence of the next corollary has been proved in [12] (see [11] for the case $a = 1$).

Corollary 4.3. *Let $n \geq |d|$ then*

$$\sum_{k=0}^{n-1} q^k \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q \equiv \left(\frac{n-|d|}{3}\right) q^{\frac{3}{2}r(r+1)+|d|(2r+1)} \pmod{\Phi_n(q)} \tag{4.4}$$

where $r = \lfloor 2(n-|d|)/3 \rfloor$. Moreover for $a > 0$ and for any prime p then

$$\sum_{k=0}^{p^a-1} \binom{2k}{k+d} \equiv \left(\frac{p^a-|d|}{3}\right) \pmod{p}.$$

Proof. For $n \geq |d|$ we use (4.3) and, since by (3.2) $\begin{bmatrix} 2n \\ k \end{bmatrix}_q$ is 0 modulo $\Phi_n(q)$ unless $k = 0, n, 2n$, then

$$\sum_{k=0}^{n-1} q^k \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q \equiv q^{\frac{1}{3}\left(2n^2 - n\left(\frac{n-|d|}{3}\right) - 2d^2 - 1\right)} \left(\frac{n-|d|}{3}\right) \begin{bmatrix} 2n \\ 0 \end{bmatrix}_q \pmod{\Phi_n(q)}$$

and the result follows. For the p-congruence, let $q = 1$ and $n = p^a$ in (4.4) and use (3.1). □

5 A dual of Greene-Krammer's identity

By Corollary 4.3, if we take $d = 0$ we have that for any prime p

$$1 + 2\sum_{k=1}^{p^a-1} \binom{2k-1}{k} = \sum_{k=0}^{p^a-1} \binom{2k}{k} \equiv \left(\frac{p^a}{3}\right) \pmod{p}.$$

The analogy mentioned at the beginning guided us to the following statement.

Theorem 5.1. *Let $q = e^{2\pi m i/n}$ with $\gcd(n, m) = 1$ then*

$$1 + 2\sum_{k=1}^{n-1} q^k \begin{bmatrix} 2k-1 \\ k \end{bmatrix}_q = \begin{cases} \left(\frac{m}{3}\right) i\sqrt{3} & \text{if } 3 \mid n \\ \left(\frac{n}{3}\right) & \text{otherwise} \end{cases}.$$

Proof. We first note that, by (2.3),

$$q^k \begin{bmatrix} 2k-1 \\ k \end{bmatrix}_q = q^{k^2} \begin{bmatrix} 2k-1 \\ k \end{bmatrix}_{1/q} = \operatorname{conj}\left(q^{-k^2} \begin{bmatrix} 2k-1 \\ k \end{bmatrix}_q \right)$$

where $\operatorname{conj}(z)$ is the complex conjugate of $z \in \mathbb{C}$.
Since $\Phi_n(q) = 0$, by (3.4) we have that

$$q^{-k^2} \begin{bmatrix} 2k-1 \\ k \end{bmatrix}_q = (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q.$$

Hence

$$\begin{aligned} 1 + 2\sum_{k=1}^{n-1} q^k \begin{bmatrix} 2k-1 \\ k \end{bmatrix}_q &= \operatorname{conj}\left(1 + 2\sum_{k=1}^{n-1} q^{-k^2} \begin{bmatrix} 2k-1 \\ k \end{bmatrix}_q \right) \\ &= \operatorname{conj}\left(-1 + 2\sum_{k\geq 0} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \right) \\ &= -1 + 2(-1)^n \left(\frac{n+1}{3}\right) q^{-\frac{1}{3}\binom{n}{2}} \end{aligned}$$

and the result is easily deduced. ∎

6 q-Catalan congruences

Theorem 6.1. *For $n > 0$*

$$\sum_{k=0}^{n-1} q^k C_k^q \equiv \begin{cases} q^{\lfloor n/3 \rfloor} & \text{if } n = 0, 1 \pmod 3 \\ -1 - q^{(2n-1)/3} & \text{if } n = 2 \pmod 3 \end{cases} \pmod{\Phi_n(q)}. \tag{6.1}$$

Proof. Since

$$C_n^q = \begin{bmatrix} 2n \\ n \end{bmatrix}_q - q \begin{bmatrix} 2n \\ n+1 \end{bmatrix}_q$$

by using (4.4) for $d = 0$ and for $d = 1$ we obtain

$$\sum_{k=0}^{n-1} q^k C_k^q \equiv q^{\frac{1}{3}(2n^2 - n\left(\frac{n}{3}\right) - 1)} \left(\frac{n}{3}\right) - q^{\frac{1}{3}(2n^2 - n\left(\frac{n-1}{3}\right))} \left(\frac{n-1}{3}\right) \pmod{\Phi_n(q)}.$$

Finally we proceed by cases on n modulo 3 and the proof is complete. ∎

The p-congruence of the next corollary has been proved in [12] (see [11] for the case $a = 1$).

Corollary 6.2. *Let $q = e^{2\pi mi/n}$ with $\gcd(n, m) = 1$. If $3 \mid n$ then*

$$\sum_{k=0}^{n-1} q^k C_k^q = \frac{1}{2} \left(i\sqrt{3} \left(\frac{m}{3}\right) - 1 \right)$$

Moreover, for any prime p and for $a > 0$

$$\sum_{k=0}^{p^a-1} C_k \equiv \frac{1}{2} \left(3 \left(\frac{p^a}{3}\right) - 1 \right) \pmod p.$$

Proof. If $3 \mid n$ then

$$\sum_{k=0}^{n-1} q^k C_k^q = q^{n/3} = e^{2\pi mi/3} = \frac{1}{2} \left(i\sqrt{3} \left(\frac{m}{3}\right) - 1 \right).$$

As regards the p-congruence, let $q = 1$ and $n = p^a$ in (6.1) and use (3.1). ∎

Theorem 6.3. *For $n > 0$*

$$\sum_{k=0}^{n-1} (-1)^k q^{-\binom{k}{2}} C_k^q \equiv F_n^q(q) + F_{n+2}^q(1) - 2 \pmod{\Phi_n(q)}. \tag{6.2}$$

and

$$F_n^q(q) + F_{n+2}^q(1) \equiv \begin{cases} (-1)^{r(n)} \left(q^{\frac{r(n)(n-1)}{2}} + q^{\frac{r(n)(n+1)}{2}} \right) & \text{if } n \equiv 0, 2, 3 \pmod 5 \\ (-1)^{r(n)} \left(q^{\frac{r(n)(n-2)}{2}} + q^{\frac{r(n)n}{2}} + q^{\frac{r(n)(n+2)}{2}} \right) & \text{if } n \equiv 1, 4 \pmod 5 \end{cases}$$

where $r(n) = round(n/5) = \lfloor n/5 + \frac{1}{2} \rfloor$.

Proof. By (3.5) and (3.6)

$$\begin{aligned}
(-1)^k q^{-\binom{k}{2}} C_k^q &\equiv (-1)^k q^{\frac{-k^2+k}{2}} \begin{bmatrix} 2k \\ k \end{bmatrix}_q - (-1)^k q^{\frac{-k^2+k+2}{2}} \begin{bmatrix} 2k \\ k+1 \end{bmatrix}_q \\
&\equiv q^{k^2+k} \begin{bmatrix} n-1-k \\ k \end{bmatrix}_q + q^{(k+1)^2} \begin{bmatrix} n-1-(k+1) \\ k+1 \end{bmatrix}_q \\
&\quad -[k=n-1] \pmod{\Phi_n(q)}.
\end{aligned}$$

Hence by applying (4.1) both for $a=0$ and $a=1$ we get

$$\begin{aligned}
\sum_{k=0}^{n-1} (-1)^k q^{-\binom{k}{2}} C_k^q &\equiv \sum_{k\geq 0}^{n-1} q^{k^2+k} \begin{bmatrix} n-1-k \\ k \end{bmatrix}_q + \sum_{k\geq 0}^{n-1} q^{(k+1)^2} \begin{bmatrix} n-k \\ k+1 \end{bmatrix}_q - 1 \\
&\equiv \sum_{k\geq 0} q^{k^2+k} \begin{bmatrix} n-1-k \\ k \end{bmatrix}_q + \sum_{k\geq 0}^{n-1} q^{k^2} \begin{bmatrix} n+1-k \\ k \end{bmatrix}_q - 2 \\
&\equiv F_n^q(q) + F_{n+2}^q(1) - 2 \pmod{\Phi_n(q)}.
\end{aligned}$$

By (4.1) for $a=1$ we find

$$F_n^q(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j(5j-3)}{2}} \begin{bmatrix} n \\ \lfloor \frac{n+2-5j}{2} \rfloor \end{bmatrix}_q .$$

Thus, by proceeding by cases on n modulo 5 we obtain

$$F_n^q(q) \equiv \begin{cases} (-1)^{r(n)} q^{\frac{(n+2)(n-1)}{10}} & \text{if } n = 1,3 \pmod 5 \\ (-1)^{r(n)} q^{\frac{(n+1)(n-2)}{10}} & \text{if } n = 2,4 \pmod 5 \\ 0 & \text{if } n = 0 \pmod 5 \end{cases} \pmod{\Phi_n(q)}$$

Similarly by (4.1) for $a=0$

$$F_{n+2}^q(1) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j(5j+1)}{2}} \begin{bmatrix} n+1 \\ \lfloor \frac{n+1-5j}{2} \rfloor \end{bmatrix}_q$$

and

$$F_{n+2}^q(1) \equiv \begin{cases} (-1)^{r(n)} \left(q^{\frac{r(n)(n+1)}{2}} + q^{\frac{r(n)(n-1)}{2}} \right) & \text{if } n = 0 \pmod 5 \\ (-1)^{r(n)} \left(q^{\frac{r(n)n}{2}} + q^{\frac{r(n)(n-2)}{2}} \right) & \text{if } n = 1 \pmod 5 \\ (-1)^{r(n)} q^{\frac{r(n)(n-1)}{2}} & \text{if } n = 2 \pmod 5 \\ (-1)^{r(n)} q^{\frac{r(n)(n+1)}{2}} & \text{if } n = 3 \pmod 5 \\ (-1)^{r(n)} \left(q^{\frac{r(n)n}{2}} + q^{\frac{r(n)(n+2)}{2}} \right) & \text{if } n = 4 \pmod 5 \end{cases} \pmod{\Phi_n(q)}$$

□

The p-congruence of the next corollary has been proved in [13].

Corollary 6.4. *Let* $q = e^{2\pi m i/n}$ *with* $\gcd(n,m)=1$*. If* $5 \mid n$ *then*

$$\sum_{k=0}^{n-1}(-1)^k q^{-\binom{k}{2}} C_k^q = \frac{1}{2}\left(\sqrt{5}\left(\frac{m}{5}\right)-3\right).$$

Moreover, for $a>0$ *and for any prime* p

$$\sum_{k=0}^{p^a-1}(-1)^k C_k \equiv \frac{1}{2}\left(5\left(\frac{p^a}{5}\right)-3\right) \pmod{p}.$$

Proof. If $5 \mid n$ then by (6.2):

$$\begin{aligned}\sum_{k=0}^{n-1}(-1)^k q^{-\binom{k}{2}} C_k^q &= -2+(-1)^{n/5}\left(q^{\frac{n(n-1)}{10}}+q^{\frac{n(n+1)}{10}}\right)\\ &= -2+(-1)^{n(m+1)/5}2\,\mathrm{Re}(e^{\pi i m/5})\\ &= \frac{1}{2}\left(\sqrt{5}\left(\frac{m}{5}\right)-3\right).\end{aligned}$$

The p-congruence follows by letting $q=1$, $n=p^a$ in (6.2) and by noting that

$$(-1)^{r(p^a)} = \left(\frac{p^a}{5}\right) \qquad \text{for } p \neq 5$$

then use (3.1). ☐

7 Proof of Theorem 4.2

Let

$$S(n,d) = \sum_{k=0}^{n-1} q^k \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q.$$

The finite sum $S(n,d)$ has some interesting properties.
The first one is that

$$S(n,d) = S(n,-d)$$

which follows immediately from $\begin{bmatrix} 2k \\ k+d \end{bmatrix}_q = \begin{bmatrix} 2k \\ k-d \end{bmatrix}_q$. The second one is less trivial.

Lemma 7.1. *Let* $n \geq |d|$ *then*

$$S(n,d) - q^{4d+6}S(n,d+3) = q^d \frac{[2d+3]_q}{[2n+1]_q}\begin{bmatrix} 2n+1 \\ n+d+2 \end{bmatrix}_q - [d=-1]q^{-1} + [d=-2]q^{-3}. \quad (7.1)$$

Proof. We first consider the case when $d \geq 0$ and we prove (7.1) by induction on n.
For $n=1$ it holds. Now we are going to prove that for $n \geq 1$

$$S(n+1,d) - q^{4d+6}S(n+1,d+3) = q^d \frac{[2d+3]_q}{[2n+3]_q}\begin{bmatrix} 2n+1 \\ n+d+3 \end{bmatrix}_q.$$

Since the l.h.s. is equal to

$$S(n,d) + q^n \begin{bmatrix} 2n \\ n+d \end{bmatrix}_q - q^{4d+6} \left(S(n,d+3) + q^n \begin{bmatrix} 2n \\ n+d+3 \end{bmatrix}_q \right),$$

by the induction hypothesis, it suffices to show that

$$q^{n-d} \begin{bmatrix} 2n \\ n+d \end{bmatrix}_q - q^{n+3d+6} \begin{bmatrix} 2n \\ n+d+3 \end{bmatrix}_q = \frac{[2d+3]_q}{[2n+3]_q} \begin{bmatrix} 2n+1 \\ n+d+3 \end{bmatrix}_q - \frac{[2d+3]_q}{[2n+1]_q} \begin{bmatrix} 2n+1 \\ n+d+2 \end{bmatrix}_q$$

which holds. Since

$$q^{k+1} \begin{bmatrix} 2k \\ k+1 \end{bmatrix}_q - q^{k+3} \begin{bmatrix} 2k \\ k+2 \end{bmatrix}_q = C^q_{k+1} - C^q_k$$

then

$$qS(n,1) - q^3 S(n,2)) = C^q_n - 1$$

and (7.1) holds for $d = -1$

$$S(n,-1) - q^2 S(n,2) = S(n,1) - q^2 S(n,2) = q^{-1} C^q_n - q^{-1},$$

and for $d = -2$

$$S(n,-2) - q^{-2} S(n,1) = S(n,2) - q^{-2} S(n,1) = -q^{-3} C^q_n + q^{-3}.$$

If $d \le -3$, by letting $d' = -d - 3 \ge 0$ we get

$$\begin{aligned} S(n,d) - q^{4d+6} S(n,d+3) &= S(n,-d) - q^{4d+6} S(n,-d-3) \\ &= -q^{-4d'-6} (S(n,d') - q^{4d'+6} S(n,d'+3)) \\ &= -q^{-3d'-6} \frac{[2d'+3]_q}{[2n+1]_q} \begin{bmatrix} 2n+1 \\ n+d'+2 \end{bmatrix}_q \\ &= q^d \frac{[2d+3]_q}{[2n+1]_q} \begin{bmatrix} 2n+1 \\ n+d+2 \end{bmatrix}_q . \end{aligned}$$

□

Finally we are ready to prove Theorem 4.2.

Proof. The q-identity (4.3) is equivalent to $S(n,d) = T(n,d)$ where

$$T(n,d) = \sum_{k \ge 0} q^{6k^2 + (3+|d|)k + |d|} \begin{bmatrix} 2n \\ n+3k+|d|+1 \end{bmatrix}_q - \sum_{k \ge 1} q^{6k^2 + (3+|d|)k + |d|} \begin{bmatrix} 2n \\ n+3k+|d|+1 \end{bmatrix}_q .$$

By the previous lemma it suffices to verify it for $d = 0, 1$ (remember that $S(n,1) = S(n,-1)$). Since the proof for $d = 1$ is quite similar, we will consider only the case for $d = 0$. So

$$T(n,0) = \sum_{k=0}^{\infty} s(n,k,3,1) - \sum_{k=1}^{\infty} s(n,k,-3,-1)$$

where

$$s(n,k,a,b) = q^{6k^2 + ak} \begin{bmatrix} 2n \\ n+3k+b \end{bmatrix}_q .$$

By using the Maple package q-Zeilberger, we verified that $s(n,k,a,b)$ solves the following recurrence

$$\sum_{j=0}^{4} a_j(n,a,b)s(n+j,k,a,b) = g(n,k+1,a,b) - g(n,k,a,b)$$

where $g(n,k,a,b) = r(n,k,a,b)s(n,k,a,b)$,

$$\begin{aligned}
a_0(n,a,b) &= (1-q^{2n+1})(1-q^{2n+2})q^6 \\
a_1(n,a,b) &= -(q^{4n+7+a-4b} + q^{4n+7-a+4b} - q^{2n+4} - q^{2n+3} + q^3 + q^2 + q + 1)q^3 \\
a_2(n,a,b) &= (q^{4n+10} + q^{2n+7} + q^{2n+6} + q^4 + q^3 + 2q^2 + q + 1)q \\
a_3(n,a,b) &= -(q^{2n+6} + q^2 + 1)(q+1) \\
a_4(n,a,b) &= 1
\end{aligned}$$

and $r(n,k,a,b)$ is the rational function certificate

$$\frac{h(n,k,a,b)(1-q^{2n+1})(1-q^{2n+2})q^{4n-12k+10-a-4b}}{(1-q^{n-3k-b+3})(1-q^{n-3k-b+4})(1-q^{n-3k-b+1})(1-q^{n+3k+b+1})}$$

with

$$\begin{aligned}
h(n,k,a,b) = {} & +q^{3n+9k+3b+a+6} - q^{3n+3k+5b+9} - q^{2n+6k+2b+a+7} + q^{2n+6k+6b+7} \\
& -q^{2n+6k+2b+a+6} + q^{2n+6k+6b+6} + q^{n+3k+b+a+7} + q^{n+3k+b+a+6} \\
& -q^{2n+6k+2b+a+5} + q^{2n+6k+5+6b} - q^{n+9k+7b+4} + q^{n+3k+b+a+5} \\
& -q^{n+9k+7b+3} - q^{n+9k+7b+2} + q^{12k+8b} - q^{a+6}.
\end{aligned}$$

Hence, since $g(n,k,a,b) = r(n,k,a,b)s(n,k,a,b) = 0$ when $|3k+b| > n$,

$$\sum_{j=0}^{4} a_j(n,a,b) \sum_{k=k_0}^{\infty} s(n+j,k,a,b) = \sum_{k=k_0}^{\infty} (g(n,k+1,a,b) - g(n,k,a,b)) = -g(n,k_0,a,b).$$

Since $a_j(n,3,1) = a_j(n,-3,-1)$ then

$$\begin{aligned}
\sum_{j=0}^{4} a_j(n,3,1)T(n+j,0) &= -g(n,0,3,1) + g(n,1,-3,-1) \\
&= -r(n,0,3,1)\begin{bmatrix} 2n \\ n+1 \end{bmatrix}_q + r(n,1,-3,-1)q^3 \begin{bmatrix} 2n \\ n+2 \end{bmatrix}_q.
\end{aligned}$$

The identity $S(n,0) = T(n,0)$ holds for $n = 1,2,3,4$ by direct verification.
By induction, it holds for $n > 4$ as soon as for $n > 1$

$$\sum_{j=0}^{4} a_j(n,3,1)S(n+j,0) = -r(n,0,3,1)\begin{bmatrix} 2n \\ n+1 \end{bmatrix}_q + r(n,1,-3,-1)q^3 \begin{bmatrix} 2n \\ n+2 \end{bmatrix}_q.$$

Let $c_i(n,3,1) = \sum_{j=i}^{4} a_j(n,3,1)$ for $i = 0,1,2,3,4$. Since $c_0(n,3,1) = 0$, the r.h.s. can be simplified, and it suffices to check that

$$\sum_{i=0}^{3} c_{i+1}(n,3,1)q^{n+i}\begin{bmatrix} 2(n+i) \\ n+i \end{bmatrix}_q = -r(n,0,3,1)\begin{bmatrix} 2n \\ n+1 \end{bmatrix}_q + r(n,1,-3,-1)q^3 \begin{bmatrix} 2n \\ n+2 \end{bmatrix}_q.$$

which holds. □

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